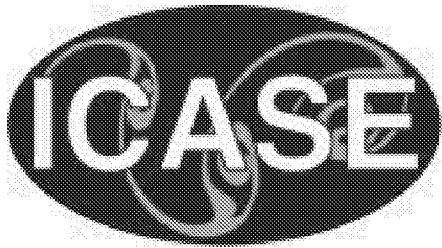


NASA/CR-2001-211060
ICASE Report No. 2001-24



Schiestel's Derivation of the Epsilon Equation and Two Equation Modeling of Rotating Turbulence

Robert Rubinstein

NASA Langley Research Center, Hampton, Virginia

Ye Zhou

Lawrence Livermore National Laboratory, Livermore, California



September 2001

The NASA STI Program Office . . . in Profile

Since its founding, NASA has been dedicated to the advancement of aeronautics and space science. The NASA Scientific and Technical Information (STI) Program Office plays a key part in helping NASA maintain this important role.

The NASA STI Program Office is operated by Langley Research Center, the lead center for NASA's scientific and technical information. The NASA STI Program Office provides access to the NASA STI Database, the largest collection of aeronautical and space science STI in the world. The Program Office is also NASA's institutional mechanism for disseminating the results of its research and development activities. These results are published by NASA in the NASA STI Report Series, which includes the following report types:

- **TECHNICAL PUBLICATION.** Reports of completed research or a major significant phase of research that present the results of NASA programs and include extensive data or theoretical analysis. Includes compilations of significant scientific and technical data and information deemed to be of continuing reference value. NASA's counterpart of peer-reviewed formal professional papers, but having less stringent limitations on manuscript length and extent of graphic presentations.
- **TECHNICAL MEMORANDUM.** Scientific and technical findings that are preliminary or of specialized interest, e.g., quick release reports, working papers, and bibliographies that contain minimal annotation. Does not contain extensive analysis.
- **CONTRACTOR REPORT.** Scientific and technical findings by NASA-sponsored contractors and grantees.

- **CONFERENCE PUBLICATIONS.** Collected papers from scientific and technical conferences, symposia, seminars, or other meetings sponsored or cosponsored by NASA.
- **SPECIAL PUBLICATION.** Scientific, technical, or historical information from NASA programs, projects, and missions, often concerned with subjects having substantial public interest.
- **TECHNICAL TRANSLATION.** English-language translations of foreign scientific and technical material pertinent to NASA's mission.

Specialized services that complement the STI Program Office's diverse offerings include creating custom thesauri, building customized data bases, organizing and publishing research results . . . even providing videos.

For more information about the NASA STI Program Office, see the following:

- Access the NASA STI Program Home Page at <http://www.sti.nasa.gov>
- Email your question via the Internet to help@sti.nasa.gov
- Fax your question to the NASA STI Help Desk at (301) 621-0134
- Telephone the NASA STI Help Desk at (301) 621-0390
- Write to:
NASA STI Help Desk
NASA Center for Aerospace Information
7121 Standard Drive
Hanover, MD 21076-1320

NASA/CR-2001-211060
ICASE Report No. 2001-24



Schiestel's Derivation of the Epsilon Equation and Two Equation Modeling of Rotating Turbulence

Robert Rubinstein

NASA Langley Research Center, Hampton, Virginia

Ye Zhou

Lawrence Livermore National Laboratory, Livermore, California

ICASE

NASA Langley Research Center

Hampton, Virginia

Operated by Universities Space Research Association



National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23681-2199

Prepared for Langley Research Center
under Contract NAS1-97046

September 2001

Available from the following:

NASA Center for AeroSpace Information (CASI)
7121 Standard Drive
Hanover, MD 21076-1320
(301) 621-0390

National Technical Information Service (NTIS)
5285 Port Royal Road
Springfield, VA 22161-2171
(703) 487-4650

SCHIESTEL'S DERIVATION OF THE EPSILON EQUATION AND TWO EQUATION MODELING OF ROTATING TURBULENCE

ROBERT RUBINSTEIN* AND YE ZHOU†

Abstract. As part of a more general program of developing multiple-scale models of turbulence, Schiestel suggested a derivation of the homogeneous part of the dissipation rate transport equation. Schiestel's approach is generalized to rotating turbulence. The resulting model reproduces the main features observed in decaying rotating turbulence.

Key words. dissipation rate equation, rotating turbulence, two-equation turbulence models, multiple-scale turbulence models

Subject classification. Fluid Mechanics

1. Introduction. The dissipation rate transport equation continues to resist systematic derivation, either from the governing equations or even from statistical closures. Much of the closure-based work is summarized in [1]; more recent work is summarized in [2]. In many respects, the most successful derivation of the ϵ transport equation is due to Schiestel [3]. Among the successes of the derivation is a rather good value $C_{\epsilon 1} = 1.5$ and the demonstration that necessarily, $C_{\epsilon 2} > C_{\epsilon 1}$.

It is well-known that the derivation of the ϵ equation in rotating turbulence encounters additional difficulties because rotation does not appear explicitly in the exact transport equation for the dissipation rate. Instead, the effect of rotation is indirect, entering only through quantities like the turbulent time-scale. In the present work, the ϵ transport equation is treated by combining Schiestel's arguments with the phenomenology for rotating turbulence of Zhou [4]. The most direct generalization of the argument of [3] leads to a rotation-sensitized ϵ equation with the same form as the standard ϵ equation, but with an increased value of $C_{\epsilon 2}$; a model of this type was proposed by Okamoto [5]. A simple modification of the argument of [3] yields instead a model of the form first proposed by Bardina *et al* [6]. The implications of these models for decaying rotating turbulence are discussed.

2. Review of Schiestel's derivation. We begin with a split-spectrum model of high Reynolds number turbulence,

$$(2.1) \quad E(\kappa) = \begin{cases} C\kappa^2 & \text{if } \kappa < \kappa_0 \\ C_K\epsilon^{2/3}\kappa^{-5/3} & \text{if } \kappa > \kappa_0 \end{cases}$$

In Eq. (2.1), κ_0 is the inverse integral scale of turbulence which marks the transition between the inertial range and the large scales. Eq. (2.1) is a special case of the models introduced in [3] in connection with multiple-scale turbulence models. This is no more than a schematic model of the actual energy spectrum; however, as stressed in [3] and [7], to derive a two-equation model, it is essential that the spectrum be

*NASA Langley Research Center, Hampton, VA 23681-2199. This research was supported by the National Aeronautics and Space Administration under NASA Contract No. NAS1-97046 while the first author was in residence at ICASE, NASA Langley Research Center, Hampton, VA 23681. R. Rubinstein acknowledges the hospitality and financial support provided by K. Hanjalic at TU Delft.

†Lawrence Livermore National Laboratory, Livermore, CA 94551. This research was partially performed under the auspices of the U.S. Department of Energy through the University of California Lawrence Livermore National Laboratory under Contract No. W-7405-Eng-48.

parametrized in some simple way. Use of a more complex model like the von Kàrmàn spectrum would lead to essentially the same results.

Denote the energy in the inertial range by

$$(2.2) \quad k = \frac{3}{2} C_K \epsilon^{2/3} \kappa_0^{-2/3}$$

and the energy in the large scales by

$$(2.3) \quad k_0 = \frac{1}{3} C \kappa_0^3$$

Assume that the spectral descriptors in Eq. (2.1) are functions of time: $\epsilon = \epsilon(t)$ and $\kappa_0 = \kappa_0(t)$. It follows from Eq. (2.2) that

$$(2.4) \quad \dot{k} = C_K (\epsilon^{-1/3} \kappa_0^{-2/3} \dot{\epsilon} - \epsilon^{2/3} \kappa_0^{-5/3} \dot{\kappa}_0)$$

This equation does not lead to the desired ϵ equation directly, because it contains the new unknown $\dot{\kappa}_0$.

To solve this problem, we postulate

$$(2.5) \quad \dot{\kappa}_0 = -\beta \frac{\epsilon}{E(\kappa_0)}$$

based on a very similar proposal in [3]. In view of Eq. (2.1), Eq. (2.5) is equivalent to

$$(2.6) \quad \dot{\kappa}_0 / \kappa_0 = -\frac{3}{2} \beta \epsilon / k$$

Then Eqs. (2.4)–(2.6) give

$$(2.7) \quad \dot{k} = \frac{2}{3} \frac{k}{\epsilon} \dot{\epsilon} - \beta \epsilon$$

which can be re-arranged as

$$(2.8) \quad \dot{\epsilon} = \frac{3}{2} \frac{\epsilon}{k} P - \frac{3}{2} (1 + \beta) \frac{\epsilon^2}{k}$$

with a rather good value for $C_{\epsilon 1}$ and a value for $C_{\epsilon 2}$ which depends on the choice of β . This result is essentially Eq. (27) of [3].

Following Reynolds [8], the constant β can be fixed by appealing to the behavior of the large scales of motion during decay. Differentiation of Eq. (2.3) gives

$$(2.9) \quad \frac{\dot{k}_0}{k_0} = 3 \frac{\dot{\kappa}_0}{\kappa_0}$$

and differentiation of Eq. (2.2) gives

$$(2.10) \quad \frac{\dot{k}}{k} = \frac{2}{3} \frac{\dot{\epsilon}}{\epsilon} - \frac{2}{3} \frac{\dot{\kappa}_0}{\kappa_0}$$

Assuming that decay is self-similar, so that

$$(2.11) \quad \frac{\dot{k}}{k} = \frac{\dot{k}_0}{k_0}$$

Eqs. (2.9)–(2.11) lead as usual to

$$(2.12) \quad \frac{\dot{\epsilon}}{\epsilon} = -\frac{11}{6} \frac{\epsilon}{k}$$

corresponding, in Eq. (2.8), to $\beta = 2/9$.

It would seem that this argument solves the problem of deriving the homogeneous ϵ transport equation, since it gives the values $C_{\epsilon 1} = 3/2$ and $C_{\epsilon 2} = 11/6$. But one can object that the assumption Eq. (2.6) is another way of stating the final result: this equation states that the integral scale κ_0^{-1} satisfies a transport equation in which the production term is absent. Indeed, writing

$$(2.13) \quad \frac{d}{dt} \frac{k^{3/2}}{\epsilon} = \frac{3}{2} \frac{k^{1/2}}{\epsilon} \dot{k} - \frac{k^{3/2}}{\epsilon^2} \dot{\epsilon}$$

and substituting

$$(2.14) \quad \begin{aligned} \dot{k} &= P - \epsilon \\ \dot{\epsilon} &= \frac{\epsilon}{k} [C_{\epsilon 1} P - C_{\epsilon 2} \epsilon] \end{aligned}$$

leads to

$$(2.15) \quad \frac{d}{dt} \frac{k^{3/2}}{\epsilon} = \left(\frac{3}{2} - C_{\epsilon 1}\right) \frac{k^{1/2}}{\epsilon} P + (C_{\epsilon 2} - \frac{3}{2}) k^{1/2}$$

which shows that the absence of a production term in the length-scale transport equation is equivalent to $C_{\epsilon 1} = 3/2$.

The injection of energy at large scales can certainly cause the integral scale to increase; at the same time, turbulence production might be expected to it to decrease through the enhancement of small scales. Eq. (2.5) states the dominance of the first process over the second. Although the validity of this approximation is uncertain, the success of the argument is undeniable, and it seems reasonable to ask what conclusions will result if the same argument is applied to another problem.

3. Rotating turbulence. To derive an ϵ equation for rotating turbulence, we will combine the arguments of the previous section with Zhou's phenomenological model of rotating turbulence [4]. Briefly, this model postulates that strong rotation replaces the nonlinear time scale k/ϵ by the inverse rotation rate Ω^{-1} ; closure theories lead to

$$(3.1) \quad \epsilon \sim \kappa^4 T E(\kappa)^2$$

where by hypothesis, $T \propto \Omega^{-1}$, hence

$$(3.2) \quad E(\kappa) = C_K^\Omega \sqrt{\epsilon \Omega} \kappa^{-2}$$

For notational simplicity, Ω will denote twice the absolute value of the rotation rate throughout.

Adding a model for the large scales, we obtain the analog of the split-spectrum model of Eq. (2.1) for rotating turbulence,

$$(3.3) \quad E(\kappa) = \begin{cases} C \kappa^2 & \text{if } \kappa < \kappa_0 \\ C_K^\Omega \sqrt{\epsilon \Omega} \kappa^{-2} & \text{if } \kappa > \kappa_0 \end{cases}$$

Again, we have the energy of the large scales,

$$(3.4) \quad k_0 = \frac{1}{3} C \kappa_0^3$$

and the inertial range energy

$$(3.5) \quad k = C_K^\Omega \sqrt{\epsilon \Omega} \kappa_0^{-1}$$

Note that the definition of the integral scale implied by Eq. (3.5) differs from the non-rotating result Eq. (2.2).

Following Schiestel, we differentiate Eq. (3.5) to obtain

$$(3.6) \quad \frac{\dot{k}}{k} = \frac{1}{2} \frac{\dot{\epsilon}}{\epsilon} - \frac{\dot{\kappa}_0}{\kappa_0}$$

As before, we must specify an equation for the inverse integral scale κ_0 in order to complete the model.

The simplest possibility is to retain Eq. (2.5). In this case, substitution of the rotation-modified spectrum Eq. (3.2) again leads to Eq. (2.6), but with a new constant of proportionality,

$$(3.7) \quad \dot{\kappa}_0/\kappa_0 = -\gamma\epsilon/k$$

Following the previous steps, we find instead of Eq. (2.8)

$$(3.8) \quad \dot{\epsilon} = 2\frac{\epsilon}{k}P - (2 + 2\gamma)\frac{\epsilon^2}{k}$$

with the definite prediction that $C_{\epsilon 1} = 2$ and $C_{\epsilon 2} > 2$.

The constant γ can be evaluated by assuming that the constant β in Eq. (2.5) is independent of rotation. Tentatively accepting the non-rotating result $\beta = 2/9$ suggested earlier, Eq. (2.5) with the rotation-dependent energy spectrum Eq. (3.2) leads to $\gamma = 2/9$ and to the value $C_{\epsilon 2} = 22/9$. In decaying rotating turbulence, Eq. (3.8) predicts power-law decay in time, but with a smaller exponent than non-rotating turbulence: indeed, following [8], we have

$$(3.9) \quad k(t) \sim t^{-1/(C_{\epsilon 2}-1)}$$

and the increase in $C_{\epsilon 2}$ due to rotation from 11/6 to 22/9 implies a reduction in the decay rate.

The model of rotating decaying turbulence implied by Eq. (3.8) has been advocated, for example in [5], and more recently in [9]. The value $C_{\epsilon 2} = 22/9$ in rotating turbulence can be compared to the values $C_{\epsilon 2} \approx 2.8$ recommended in [5] and $C_{\epsilon 2} = 8/3$ suggested in [9].

However, the available data is also consistent with the conclusion that in rotating turbulence, energy transfer is suppressed completely, and energy becomes trapped in the largest scales of motion, where it undergoes purely viscous decay. This picture, which is inconsistent with any kind of power-law decay, is advocated for example by [10] and [11]. Which description of decaying rotating turbulence is correct remains controversial; for now, we would like to explore some models which are consistent with the second viewpoint.

The derivation of Eq. (3.2) assumes that the time-scale in strongly rotating turbulence is the inverse rotation rate. This idea suggests replacing Eq. (2.6) by

$$(3.10) \quad \frac{\dot{\kappa}_0}{\kappa_0} = -\gamma'\Omega$$

in the strong rotation limit. Eqs. (3.6) and (3.10) yield the ϵ equation in the form

$$(3.11) \quad \dot{\epsilon} = 2\frac{\epsilon}{k}(P - \epsilon) - \gamma'\Omega\epsilon$$

The rotation dependence found in Eq. (3.11) coincides with that of the well-known Bardina model [6]; we argued previously [1] for the strong rotation limit of this model on the basis of simplified closure arguments. Integration of the Bardina model for decaying turbulence in the strong rotation limit gives the results that ϵ decays exponentially in time, but that the kinetic energy approaches a constant; if viscosity is included in the analysis, then the kinetic energy undergoes purely viscous decay.

Although these conclusions are consistent with numerical and experimental observations [10], the assumption Eq. (3.10) underlying the present derivation has the consequence that the integral scale grows exponentially. This was cited in [9] as evidence against the Bardina model itself, although [11] argued that quite different two-point behavior can be consistent with the same single-point model.

The difficulty is not so much with Schiestel's formalism, but with applying Eq. (3.10), an isotropic result, to rotating turbulence. In rotating turbulence, the Taylor-Proudman theorem forces the large scales of motion to be nearly two-dimensional. Consequently, the integral scales parallel and perpendicular to the rotation axis are unequal [10].

It is rather difficult to capture this effect in any isotropic model. But suppose that we combine Eqs. (3.6) and (2.9) to give

$$(3.12) \quad \frac{\dot{k}}{k} = \frac{1}{2} \frac{\dot{\epsilon}}{\epsilon} - \frac{1}{3} \frac{\dot{k}_0}{k_0}$$

and simply postulate the large rotation limit of Eq. (3.11) for decaying turbulence

$$(3.13) \quad \frac{\dot{\epsilon}}{\epsilon} = -\gamma' \Omega$$

Then we obtain

$$(3.14) \quad \frac{\dot{k}}{k} = -\frac{1}{2} \gamma' \Omega - \frac{1}{3} \frac{\dot{k}_0}{k_0}$$

or equivalently,

$$(3.15) \quad k k_0^{1/3} = k(0) k_0(0)^{1/3} e^{-\gamma' \Omega t/2}$$

instead of the self-similarity postulate Eq. (2.11) for non-rotating turbulence. Unlike the argument leading to Eq. (3.8), which like the derivation for isotropic turbulence assumes that the energy decay of the large scales and the inertial range scales is linked by self-similarity, the present derivation instead allows the dynamics of the large scales and the inertial range scales to be different.

The problem of decaying rotating turbulence is defined by the energy equation together with Eq. (3.13) and either Eq. (3.14) or Eq. (3.15). Numerical integration will be required to solve these equations in general, but it is evident that these equations are consistent with the limits

$$(3.16) \quad \begin{aligned} \epsilon &= 0 \\ k &= 0 \end{aligned}$$

while

$$(3.17) \quad \begin{aligned} k_0 &= \text{const.} \\ \kappa_0 &= \text{const.} \end{aligned}$$

Thus, the kinetic energy in the inertial range vanishes, the energy transfer vanishes, but the kinetic energy in the large scales and the integral scale both approach constants in the absence of viscosity.

Let us summarize the differences between the two dynamic descriptions of rotating decay. Power-law decay, but with a reduced exponent, follows if the decay of both the large-scale energy and the inertial range energy is linked through the self-similarity assumption Eq. (2.11). The alternative description, which leads instead to Eqs. (3.16) and (3.17) allows the large-scale and inertial range energies to evolve independently. The argument also implies that in the long-time limit, viscous dissipation and energy transfer are unequal: energy transfer can vanish, but viscous dissipation is always nonzero.

4. Conclusions. Schiestel's derivation of the ϵ transport equation has been generalized to rotating turbulence. By assuming that the basic scale relationship Eq. (2.5) applies to both non-rotating and rotating turbulence, we are led to the ϵ equation in the form Eq. (3.8). This equation implies algebraic decay in time of decaying rotating turbulence with a smaller decay rate than non-rotating turbulence. Replacing Eq. (3.8) with the rotation-dependent hypothesis Eq. (3.10) leads essentially to the Bardina model, which implies a completely different description of rotating decay: the nonlinear energy transfer vanishes and in the absence of viscous effects, energy approaches a constant. By ignoring the two-dimensionality and rotation-independence of the large scales, this argument leads to an incorrect description of the integral scale in decaying rotating turbulence. By modifying Schiestel's argument, the Bardina model is shown to be consistent with saturation of the integral scale.

REFERENCES

- [1] R. RUBINSTEIN AND YE ZHOU, *Analytical theory of the destruction terms in dissipation rate transport equations*, Phys. Fluids **8** (1996), p. 3172.
- [2] A. CADIOU AND K. HANJALIĆ, *One-point multi-scale turbulence closure derived from a spectral description*, TU Delft Report APTF R/99-11, 1999.
- [3] R. SCHIESTEL, *Multiple-time-scale modeling of turbulent flows in one point closures*, Phys. Fluids **30** (1987), p. 722.
- [4] YE ZHOU, *A phenomenological treatment of rotating turbulence*, Phys. Fluids **7** (1995), p. 2092.
- [5] M. OKAMOTO, *Theoretical turbulence modeling of homogeneous decaying flow in a rotating frame*, J. Phys. Soc. Japan **64** (1995), p. 2854.
- [6] J. BARDINA, J. H. FERZIGER, AND R. S. ROGALLO, *Effect of rotation on isotropic turbulence: Computation and modeling*, J. Fluid Mech. **154** (1985), p. 321.
- [7] T. CLARK AND C. ZEMACH, *A spectral model applied to homogeneous turbulence*, Phys. Fluids **7** (1995), p. 1674.
- [8] W. C. REYNOLDS, *Computation of turbulent flows*, Ann. Rev. Fluid Mech. **8** (1976), p. 183.
- [9] V. M. CANUTO AND M. S. DUBOVIKOV, *A dynamical model of turbulence V: The effect of rotation*, Phys. Fluids **9** (1997), p. 2132.
- [10] C. CAMBON AND J. SCOTT, *Linear and nonlinear models of anisotropic turbulence*, Ann. Rev. Fluid Mech. **31** (1999), p. 1.
- [11] YE ZHOU, C. G. SPEZIALE, AND R. RUBINSTEIN, *Some remarks concerning recent work on rotating turbulence*, Phys. Fluids **10** (1998), p. 3242.

REPORT DOCUMENTATION PAGE			Form Approved OMB No. 0704-0188	
Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188), Washington, DC 20503.				
1. AGENCY USE ONLY(Leave blank)		2. REPORT DATE September 2001		3. REPORT TYPE AND DATES COVERED Contractor Report
4. TITLE AND SUBTITLE Schiestel's derivation of the epsilon equation and two equation modeling of rotating turbulence			5. FUNDING NUMBERS C NAS1-97046 WU 505-90-52-01	
6. AUTHOR(S) Robert Rubinstein and Ye Zhou				
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) ICASE Mail Stop 132C NASA Langley Research Center Hampton, VA 23681-2199			8. PERFORMING ORGANIZATION REPORT NUMBER ICASE Report No. 2001-24	
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) National Aeronautics and Space Administration Langley Research Center Hampton, VA 23681-2199			10. SPONSORING/MONITORING AGENCY REPORT NUMBER NASA/CR-2001-211060 ICASE Report No. 2001-24	
11. SUPPLEMENTARY NOTES Langley Technical Monitor: Dennis M. Bushnell Final Report To appear in Computers and Mathematics with Applications.				
12a. DISTRIBUTION/AVAILABILITY STATEMENT Unclassified-Unlimited Subject Category 34 Distribution: Nonstandard Availability: NASA-CASI (301) 621-0390			12b. DISTRIBUTION CODE	
13. ABSTRACT (Maximum 200 words) As part of a more general program of developing multiple-scale models of turbulence, Schiestel suggested a derivation of the homogeneous part of the dissipation rate transport equation. Schiestel's approach is generalized to rotating turbulence. The resulting model reproduces the main features observed in decaying rotating turbulence.				
14. SUBJECT TERMS dissipation rate equation, rotating turbulence, two-equation turbulence models, multiple-scale turbulence models			15. NUMBER OF PAGES 11	
			16. PRICE CODE A03	
17. SECURITY CLASSIFICATION OF REPORT Unclassified	18. SECURITY CLASSIFICATION OF THIS PAGE Unclassified	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT	